STOCHASTIC VARIATIONAL PRINCIPLES FOR DISSIPATIVE EQUATIONS

Tudor S. Ratiu
Department of Mathematics
Jiao Tong University, Shanghai, China

Xin Chen (SJTU) and Ana Bela Cruzeiro (ISTL)

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SEMIDIRECT PRODUCT THEORY

$\rho : G \to \mathrm{Aut}(V)$ denote a *right* Lie group representation. Form the semidirect product $S = G \circledS V$ whose group multiplication is

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)).$$

The Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ of S has bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{d}{dt}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

If $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given by

$$a\xi := \left.\frac{d}{dt}\right|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi\rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V,$$

$\langle \cdot, \cdot\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot, \cdot\rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

Lagrangian semidirect product theory

- $L : TG \times V^* \to \mathbb{R}$ which is right G-invariant.

- So, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where $G_{a_0} := \{g \in G \mid \rho_g^* a_0 = a_0\}$.

- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_g R_{g^{-1}}(v_g), \rho_g^*(a_0)) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t),$$

with initial condition $a(0) = a_0$. Solution is $a(t) = \rho^*_{g(t)}(a_0)$.

i *With a_0 held fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations $(\delta\xi, \delta a)$ of the form

$$\delta\xi = \frac{\partial\eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta,$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv *The Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a.$$

Example in Euclidean space $\mathbb{R}^n$

DETERMINISTIC: Vector field u (velocity), $g(t)$ an integral curve

$$dg(t,x) = u(t, g(t,x))dt, \qquad g(0,x) = x.$$

RANDOM: Perturb the paths $g(t)$ by a Brownian motion $\sigma W_\omega(t)$, $\sigma \in \mathbb{R}$, no longer differentiable, but can still write

$$dg_\omega(t,x) = \sigma dW(t) + u(t, g_\omega(t,x))dt, \;\; g_\omega(0,x) = x, \;\; \omega\text{-almost everywhere}$$

as long as we interpret the differential d in sense of the Itô calculus.

Heuristically, the increments $g_\omega(t+\epsilon, x) - g_\omega(t,x)$ have normal distribution with mean $\epsilon u(t, g_\omega(t,x))$ and variance $\epsilon\sigma^2$.

Dynamically, the paths g follow "in the mean" the directions determined by the vector field u (the drift) but are subjected to a dispersion due to the Brownian motion effect. More precisely, we have, almost-everywhere in ω,

$$u(t, g_\omega(t,x)) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(\mathbb{E}_t g_\omega(t+\epsilon, x) - g_\omega(t,x)\Big)$$

where $\mathbb{E}_t$ denotes conditional expectation given the past at time t. So, u still corresponds to a (generalized) derivative.

The generator of the process $g_\omega(t)$ is the operator defined by $Lf := \frac{\sigma^2}{2}\Delta f + (u\cdot\nabla)f$, i.e., we have

$$\mathbb{E}[f(g_\omega(t,x))] = f(x) + \mathbb{E}\int_0^t [Lf(g_\omega(s,x))]ds,$$

for every (compactly supported) smooth function f, where $\mathbb{E}$ denotes the expectation operator on the probability space whose points are ω. The Laplacian term accounts for the dispersion. In this sense, L deforms the time derivative along the classical flow with velocity field u.

Using such stochastic processes as "Lagrangian paths" one can derive variational principles extending those of classical mechanics. The critical points of action functionals, involving the above mentioned Lagrangians, solve almost surely the equations of motion. Those reduce to classical (Euler-Lagrange) equations in the limit $\sigma = 0$, by construction. It follows, in particular, that as in this classical limit case, the Lagrangian encodes most of the qualitative properties of the (stochastic) system.

Cruzeiro: the Navier-Stokes equation was obtained in this manner, in the usual Eulerian (spatial) description, via a stochastic Euler-Poincaré reduction on the group of volume preserving diffeomorphisms (three papers with Arnaudon, Chen, Cipriano). "The L^2-geodesics for the averaged stochastic fluid particle paths are the solutions of the Navier-Stokes equations." The Lagrangian variables correspond to semimartingales.

GOAL: Do the same thing for fluids with advected quantitities. We know that semidirect product theory will appear.

NOTATIONS AND CONVENTIONS

$\mathbb{R}^+ := [0, \infty[$, $(\Omega, \mathcal{P}, P)$ probability space.

Non-decreasing filtration $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on the probability space:

- $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ given family of sub-σ-algebras of $\mathcal{P}$
- non-decreasing: $\mathcal{P}_s \subseteq \mathcal{P}_t$ if $0 \leq s \leq t$
- right-continuous: $\cap_{\epsilon>0}\mathcal{P}_{t+\epsilon} = \mathcal{P}_t$, $\forall t \in \mathbb{R}^+$.

A stochastic process $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $(\mathcal{P}_t)$-*adapted* if $X(t)$ is $\mathcal{P}_t$-measurable for every t. Typically, filtrations describe the past history of a process: one starts with a process X and defines $\mathcal{P}_t := \cap_{\epsilon>0}\sigma\{X(s), \epsilon \leq s \leq t\}$. Then the process X will be automatically $(\mathcal{P}_t)$-adapted.

$\mathbb{E}$ denotes the *expectation* of a random variable

$\mathbb{E}_s(M(t,\omega)) := \mathbb{E}(M(t,\omega)|\mathcal{P}_s)$, for each $s \geq 0$, is the *conditional expectation* of the random variable $M_\omega(t)$, $t > s$, relative to the σ-algebra $(\mathcal{P}_s)$, i.e., $\Omega \ni \omega \mapsto \mathbb{E}_s(M_\omega(t)) \in \mathbb{R}$ is a $\mathcal{P}_s$-measurable function satisfying

$$\int_A \mathbb{E}_s(M(t,\omega))dP(\omega) = \int_A M(t,\omega)dP(\omega), \quad \forall A \in \mathcal{P}_s.$$

A stochastic process $M : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is a *martingale* if
(i) $\mathbb{E}|M(t,\omega)| < \infty$ for all $t \geq 0$;
(ii) $M(t,\omega)$ is $(\mathcal{P}_t)$-adapted;
(iii) $\mathbb{E}_s(M(t,\omega))) = M(s,\omega)$ a.s. for all $0 \leq s < t$.
Condition **(iii)** is equivalent to $\mathbb{E}((M_\omega(t) - M_\omega(s))\chi_A) = 0$, $\forall A \in \mathcal{P}_s$, $\forall t, s \in \mathbb{R}$ satisfying $t > s \geq 0$; χ_A characteristic function of set A.

Work only with processes defined on compact time intervals $[0,T]$, continuous in t for almost all $\omega \in \Omega$, i.e., *continuous processes*.

If a martingale M is continuous and $\mathbb{E}(M(t,\cdot)^2) < \infty$, $\forall t \geq 0$, then M has a *quadratic variation* $\{[\![M,M]\!]_t,\ t \in [0,T]\}$ if $M(t)^2 - [\![M,M]\!]_t$ is a martingale, and $[\![M,M]\!]_t$ is a continuous, $\mathcal{P}_t$-adapted, a.s. non-decreasing process with $[\![M,M]\!]_0 = 0$. Such a process is unique and coincides with the following limit (convergence in probability),

$$\lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} (M(t_{i+1}) - M(t_i))^2;$$

σ_n is a partition of $[0,t]$ and the mesh converges to zero as $n \to \infty$. Def. of the quadratic variation requires only right-continuity of M.

M, N martingales, same assumptions; their *covariation* is

$$[\![M,N]\!]_t := \lim_{n\to\infty} \sum_{t_i,t_{i+1}\in\sigma_n} (M(t_{i+1}) - M(t_i))(N(t_{i+1}) - N(t_i)),$$

which extends the notion of quadratic variation. Clearly,

$$2[\![M,N]\!]_t = [\![M+N,M+N]\!]_t - [\![M,M]\!]_t - [\![N,N]\!]_t.$$

Stopping time: random variable $\tau : \Omega \to \mathbb{R}^+$ such that

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{P}_t, \forall t \geq 0.$$

Local martingale: stochastic process M for which $\exists$ sequence of stopping times $\{\tau_n, n \geq 1\}$, such that $\lim_{n\to\infty} \tau_n = \infty$ a.s., and $M^n(t) := M(t \wedge \tau_n)$ is a square integrable martingale for all $n \geq 1$, where $t \wedge \tau_n := \min(t, \tau_n)$. Define $[\![M, M]\!]_t := [\![M^n, M^n]\!]_t$ if $t \leq \tau_n$.

Real-valued Brownian motion: continuous martingale $W(t)$, $t \in [0, T]$, such that $W^2(t) - t$ is a martingale $\iff [\![W, W]\!]_t = t$.

Semimartingale: stochastic process $X : \Omega \times [0, T] \to \mathbb{R}$ such that

$$X(t) = X(0) + M(t) + A(t), \quad \forall t \geq 0,$$

where M is a local martingale with $M(0) = 0$ and A is a càdlàg (A is right-continuous with left limits at each $t \geq 0$) adapted process of locally bounded variation with $A(0) = 0$ a.s. We consider only processes that are continuous in time.

Local semimartingale: same as above with M a local martingale and A a locally bounded variation process. Define $[\![X, X]\!]_t := [\![M, M]\!]_t$.

Martingales (hence Brownian motion) are not a.s. t-differentiable (unless they are constant), so cannot integrate with respect to martingales as one does with respect to functions of bounded variation. Needed stochastic integrals: the Itô and the Stratonovich integrals.

If X and Y are continuous real-valued semimartingales such that

$$\mathbb{E}\left(\int_0^T |X(t)|^2 dt + \int_0^T |Y(t)|^2 dt\right) < \infty,$$

the *Itô stochastic integral* of $X(t)$ on $[0,t]$, $0 < t \leq T$, with respect to Y is defined as the limit in probability (if limit exists) of the sums

$$\int_0^t X(s) dY(s) := \lim_{n\to\infty} \sum_{t_i, t_{i+1} \in \sigma_n} X(t_i)(Y(t_{i+1}) - Y(t_i));$$

σ_n is a partition of $[0,t]$ with mesh converging to zero as $n \to \infty$.

If Y is a martingale such that $\mathbb{E}\left(\int_0^T |X(t)|^2 d[\![Y,Y]\!]_t\right) < \infty$, then $\int_0^t X(s) dY(s)$, $t \in [0,T]$, is also a martingale.

The *Stratonovich stochastic integral* is defined by

$$\int_0^t X(s) \circ dY(s) := \lim_{n\to\infty} \sum_{t_i, t_{i+1} \in \sigma_n} \frac{(X(t_i) + X(t_{i+1}))}{2} (Y(t_{i+1}) - Y(t_i))$$

whenever this limit exists.

These integrals do not coincide in general, even if X is a continuous process, due to the lack of differentiability of the paths of Y. The Itô and the Stratonovich integrals are related by

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2} \int_0^t d[\![X, Y]\!]_s$$

Itô's formula: for any $f \in C^2(\mathbb{R})$,

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s)) dX(s) + \frac{1}{2} \int_0^t f''(X(s)) d[\![X, X]\!]_s$$

For Stratonovich integrals, this formula is:

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s)) \circ dX(s)$$

Stratonovich integral: standard differential calculus rules apply, works on manifolds.

Itô integral with respect to a martingale M is again a martingale, a very important property. For example, we have, as an immediate consequence, that $\mathbb{E}_s \int_s^t X(r) dM(r) = 0$ for all $0 \le s < t$.

Itô's formula: X a $\mathbb{R}^d$-valued semimartingale; for any $f \in C^2(\mathbb{R}^d)$,

$$f(X(t)) = f(X(0)) + \sum_{i=1}^{d} \int_0^t \partial_i f(X(s)) dX^i(s) + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial^2_{i,j}(X(s)) d[\![X^i, X^j]\!]_s$$

$$= f(X(0)) + \sum_{i=1}^{d} \int_0^t \partial_i f(X(s)) \circ dX^i(s)$$

The difference between the two integrals is given by the Hessian.

Rules for independent Brownian motions $W^1, \dots W^k$, $\iota(t) := t$:

$$d[\![W^i, W^j]\!]_t = \delta_{ij} dt, \quad d[\![W^i, \iota]\!]_t = 0, \quad d[\![\iota, \iota]\!]_t = 0, \quad \forall i, j = 1, \dots, d$$

(covariation of semimartingales determined by their martingale parts)

GENERALIZED DERIVATIVE FOR LIE GROUP VALUED SEMIMARTINGALES

G Lie group. L_g, R_g left and right translation by $g \in G$. If $v \in T_eG$, $v^L(g) := T_eL_gv$ and $v^R(g) := T_eR_gv$ are the left and right invariant vector fields on G such that $v^L(e) = v^R(e) = v$. $[v_1, v_2] := \left[v_1^L, v_2^L\right](e)$, for $v_1, v_2 \in T_eG$, defines a (left) Lie bracket on T_eG. Denote $\mathrm{ad}_u v := [u, v]$ and $\mathrm{ad}_u^* : T_e^*G \to T_e^*G$ its dual map.

Let ∇ be a left invariant linear connection on G, i.e., $\nabla_{v_1^L} v_2^L$ is a left invariant vector field, for any $v_1, v_2 \in T_eG$. Define

$$\nabla_{v_1} v_2 := \nabla_{v_1^L} v_2^L(e), \quad \forall v_1, v_2 \in T_eG.$$

Left invariant ∇ is torsion free $\iff$

$$\nabla_{v_1} v_2 - \nabla_{v_2} v_1 = [v_1, v_2], \quad \text{for all} \quad v_1, v_2 \in T_eG.$$

Let ∇ be a left invariant linear torsion free connection. The *Hessian* $\mathrm{Hess} f(g) : T_gG \times T_gG \to \mathbb{R}$ of $f \in C^2(G)$ at $g \in G$ is defined by

$$\mathrm{Hess} f(g)(v_1, v_2) := \tilde{v}_1\tilde{v}_2 f(g) - \nabla_{\tilde{v}_1}\tilde{v}_2 f(g), \quad v_1, v_2 \in T_gG,$$

for $\tilde{v}_i$, $i = 1, 2$, arbitrary vector fields on G such that $\tilde{v}_i(g) = v_i$. Since ∇ is torsion free, $\mathrm{Hess}\, f(g)$ is a symmetric $\mathbb{R}$-bilinear form on each T_gG. $\mathrm{Hess} f = \nabla^2 f = \nabla \mathbf{d} f$.

A G-valued semimartingale is called a ∇-*(local) martingale* if

$$t \longmapsto f(g_\omega(t)) - f(g_\omega(0)) - \frac{1}{2}\int_0^t \mathrm{Hess} f(g_\omega(s)) d[\![g_\omega, g_\omega]\!]_s ds$$

is a real-valued (local) martingale, $\forall f \in C^2(G)$, where $[\![g_\omega, g_\omega]\!]_t$ is the quadratic variation of g_ω. If G is finite dimensional, then

$$d[\![g_\omega, g_\omega]\!]_t := d\left[\!\!\left[\int_0^\cdot \mathbf{P}_s^{-1} \circ dg_\omega(s), \int_0^\cdot \mathbf{P}_s^{-1} \circ dg_\omega(s)\right]\!\!\right]_t,$$

where $\mathbf{P}_t : T_{g_\omega(0)}G \to T_{g_\omega(t)}G$ is the (stochastic) ∇-parallel translation along the (stochastic) curve $t \mapsto g_\omega(t)$. For some infinite dimensional groups G (e.g., diffeomorphisms on a torus), the quadratic variation is also well defined.

For a G-valued semimartingale $g_\omega(\cdot)$, suppose there is a, possibly random, process $(t,\omega) \mapsto \mathbf{v}(t,\omega) \in TG$, such that $\mathbf{v}_\omega(t) \in T_{g_\omega(t)}G$ a.s., and for every $f \in C^\infty(G)$, the process

$$t \longmapsto N_t^f := f(g_\omega(t)) - f(g_\omega(0))) - \frac{1}{2}\int_0^t \mathrm{Hess} f(g_\omega(s)))d[\![g_\omega, g_\omega]\!]_s - \int_0^t \langle \mathbf{d}f(g_\omega(s)), \mathbf{v}_\omega(s)\rangle\, ds \quad \text{is a real-valued local martingale.}$$

Define the ∇-*generalized derivative* of $g_\omega(t)$ by

$$\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} := \mathbf{v}_\omega(t),$$

well defined for semimartingales with values in a finite dimensional Lie group and some infinite dimensional groups.

If G finite dimensional compact, then (Emery)

$$\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} := \mathbf{P}_t\left(\lim_{\epsilon\to 0} \mathbb{E}_t\left[\frac{\eta_\omega(t+\epsilon) - \eta_\omega(t)}{\epsilon}\right]\right) \in T_{g_\omega(t)}G, \quad \text{where}$$

$$\eta_\omega(t) := \int_0^t \mathbf{P}_s^{-1} \circ dg_\omega(s) \in T_eG.$$

The conditional expectation $\mathbb{E}_t$ plays a major rôle in the definition of the ∇-generalized derivative, since it eliminates the martingale part of the semimartingale. Therefore, the velocities are given by the drift (the bounded variation part) and the diffusion part (the martingale) can be seen as a stochastic perturbation; in other words, the drift determines the directions where the particles flow, the martingale part describes their random fluctuations.

If a G-valued semimartingale $g_\omega(t)$ satisfies $\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} = 0$, then $g_\omega(t)$ is a ∇-martingale.

In Euclidean space, endowed with the standard Levi-Civita connection of the constant Riemannian metric given by the inner product, $\frac{\mathscr{D}^\nabla g_\omega(t)}{dt}$ coincides with the standard definition of the generalized derivative for Euclidean valued semimartingale (Cipriano-Cruzeiro [2007], Yasue [1981], Zambrini [2015]).

Given an $\mathbb{R}^k$-valued Brownian motion $W_\omega(t) = (W_\omega^1(t), ..., W_\omega^k(t))$, $t \in [0, T]$, vectors $H_i \in T_eG$, $1 \leq i \leq k$, and a curve $u \in C^1([0, T]; T_eG)$, we consider the following Stratonovich SDE on G:

$$\begin{cases} dg_\omega(t) = T_eL_{g_\omega(t)} \left(\sum_{i=1}^k H_i \circ dW_\omega^i(t) - \frac{1}{2}\nabla_{H_i} H_i dt + u(t)dt \right), \\ g_\omega(0) = e. \end{cases}$$

Since $d[\![(T_eL_{g_\omega(t)} H_i), W_\omega^i(t)]\!]_t = T_eL_{g_\omega(t)}(\nabla_{H_i} H_i)dt$, the formula relating the Stratonovich and Itô integrals shows that this equation is equivalent to

$$\begin{cases} dg_\omega(t) = T_eL_{g_\omega(t)} \left(\sum_{i=1}^k H_i dW_\omega^i(t) + u(t)dt \right), \\ g_\omega(0) = e. \end{cases}$$

Known: If G is a finite dimensional Lie group, there exists a unique strong solution (Ikeda-Watanabe, Emery). If G is the diffeomorphism group on a torus, a weak solution exists (Arnaudon-Chen-Cruzeiro [2014], Cipriano-Cruzeiro [2007]).

Apply Itô's formula for G-valued semimartingales (Emery for G finite dimensional, Arnaudon-Chen-Cruzeiro [2014] for the diffeomorphism group of a torus): for every $f \in C^2(G)$ we have,

$$\begin{aligned} f(g_\omega(t)) =& f(g_\omega(0)) + N_t^f + \frac{1}{2}\int_0^t \mathrm{Hess} f(g_\omega(s)) d[\![g_\omega, g_\omega]\!]_s \\ & + \int_0^t \left\langle \mathbf{d}f(g_\omega(s)), T_e L_{g_\omega(s)} u(s) \right\rangle ds, \end{aligned}$$

where N_t^f is a martingale. Actually, this last equality, valid for each $f \in C^2(G)$, is a characterization of the solution of the stochastic differential equation above, in a weak sense.

Thus, using the definition and the comments above, if G is finite dimensional, using the definition and this identity, we get

$$\frac{\mathscr{D}^\nabla g_\omega(t)}{dt} = T_e L_{g_\omega(t)} u(t).$$

Remember: $t \mapsto u(t) \in \mathfrak{g}$ is a deterministic path.

STOCHASTIC SEMIDIRECT PRODUCT EULER-POINCARÉ EQUATIONS

U vector (Banach) space, U^* dual, $\langle\cdot,\cdot\rangle_U : U^* \times U \to \mathbb{R}$ duality pairing.

G Lie group (enough: topological group, manifold, smooth left translation). T_eG its Lie algebra (ILB). U left representation space for G. So there are naturally induced left representations of G and T_eG on U and U^*. All representations are denoted by concatenation. $\langle\cdot,\cdot\rangle_{T_eG} : T_e^*G \times T_eG \to \mathbb{R}$ duality pairing.

Define the operator $\diamond : U \times U^* \to T_e^*G$ by

$$\langle a \diamond \alpha, v \rangle_{T_eG} := -\langle v\alpha, a \rangle_U = \langle \alpha, va \rangle_U, \quad v \in T_eG, \quad a \in U, \quad \alpha \in U^*.$$

$a \diamond \alpha$ is the value at (a, α) of the momentum map $U \times U^* \to T_e^*G$ of the cotangent lifted action induced by the left representation of G on U.

$\mathscr{S}(G)$ all G-valued continuous semimartingales defined for $t \in [0, T]$. Given a left invariant linear connection ∇ on G, a point $\alpha_0 \in U^*$, and a (Lagrangian) function $l : T_eG \times U^* \to \mathbb{R}$, define the *action functional* $J^{\nabla,\alpha_0,l} : \mathscr{S}(G) \times \mathscr{S}(G) \to \mathbb{R}_+$ by

$$J^{\nabla,\alpha_0,l}(g^1_\omega(\cdot), g^2_\omega(\cdot)) := \int_0^T l\left(T_{g^1_\omega(t)} L_{g^1_\omega(t)^{-1}} \frac{\mathscr{D}^\nabla g^1_\omega(t)}{dt}, \alpha(t)\right) dt,$$

where $g^1_\omega(\cdot), g^2_\omega(\cdot) \in \mathscr{S}(G)$ and

$$\alpha(t) := \mathbb{E}\left[\tilde{\alpha}_\omega(t)\right] \in U^*, \qquad \tilde{\alpha}_\omega(t) := g^2_\omega(t)^{-1}\alpha_0.$$

The first variable in l is $u(t)$, deterministic.

For every (deterministic) curve $g(\cdot) \in C^1([0,1]; T_eG)$, $g(0) = g(T) = 0$, and $\varepsilon \in [0,1)$, let $e_{\varepsilon,g}(\cdot) \in C^1([0,T]; G)$ be the unique solution of the (deterministic) time-dependent differential equation on G

$$\begin{cases} \frac{d}{dt} e_{\varepsilon,g}(t) = \varepsilon T_e L_{e_{\varepsilon,g}(t)} \dot{g}(t), \\ e_{\varepsilon,g}(0) = e. \end{cases}$$

Note that this system implies $e_{0,g}(t) = e$ for all $t \in [0, T]$.

$\Big(g^1_\omega(\cdot), g^2_\omega(\cdot)\Big) \in \mathscr{S}(G)\times\mathscr{S}(G)$ is a *critical point of* $J^{\nabla,\alpha_0,l}$ if for every (deterministic) curve $\mathfrak{g}(\cdot) \in C^1([0,1]; T_eG)$ with $\mathfrak{g}(0) = \mathfrak{g}(T) = 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} J^{\nabla,\alpha_0,l}\left(g^1_{\omega,\varepsilon,\mathfrak{g}}(\cdot), g^2_{\omega,\varepsilon,\mathfrak{g}}(\cdot)\right) = 0,$$

$$g^i_{\omega,\varepsilon,\mathfrak{g}}(t) := g^i_\omega(t) e_{\varepsilon,\mathfrak{g}}(t),\ t \in [0,T],\ i = 1,2,\ \varepsilon \in [0,1).$$

Note the particular form of these deformations in the Lie group: they correspond to developments along deterministic directions $\mathfrak{g}(t)$.

We choose two different semimartingales $(g^1_\omega(\cdot), g^2_\omega(\cdot))$ in the variational principle because, in applications, the viscosity constants in the equations for the variables in T_eG and U^* may be different.

Assume G, U are finite dimensional. Main theorem still holds in some infinite dimensional models, e.g., Diff$(\mathbb{T}^n)$.

Fix $H^m_j \in T_eG$, $m = 1,2$, $j = 1,...,k_m$, $W^m_\omega(t) = (W^{m,1}_\omega(t),...,W^{m,k_m}_\omega(t))$, $m = 1,2$, independent $\mathbb{R}^{k_m}$ valued Brownian motions (in infinite dimensions, we may not be able to fix a priori the Brownian motions in order to solve the corresponding stochastic differential equations).

Define the semimartingales $h^m_\omega(\cdot) \in \mathscr{S}(G)$, $m = 1, 2$,

$$\begin{cases} \quad dh^m_\omega(t) = T_e L_{h^m_\omega(t)} \left(\sum_{j=1}^{k_m} \left(H^m_j \circ dW^{m,j}_\omega(t) \right) + u(t)dt \right), \\ \quad h^m_\omega(0) = e, \end{cases}$$

for $u(\cdot) \in C^1([0, T]; T_eG)$. Accordingly, define

$$\alpha(t) := \mathbb{E}\Big[\tilde{\alpha}_\omega(t)\Big] \in U^*, \qquad \tilde{\alpha}_\omega(t) := h^2_\omega(t)^{-1}\alpha_0.$$

Consider deformations the form $h^i_{\omega,\varepsilon,\mathscr{g}}(t) := h^i_\omega(t)e_{\varepsilon,\mathscr{g}}(t)$, $t \in [0, T]$, $i = 1, 2$, $\varepsilon \in [0, 1)$.

Define the operator $K : T^*_eG \to T^*_eG$: if $\mu \in T^*_eG$ and $v \in T_eG$,

$$\langle K(\mu), v \rangle = -\left\langle \mu, \frac{1}{2} \sum_{j=1}^{k_1} \left(\nabla_{\mathrm{ad}_v H^1_j} H^1_j + \nabla_{H^1_j} (\mathrm{ad}_v H^1_j) \right) \right\rangle.$$

$\left(h_\omega^1(\cdot), h_\omega^2(\cdot)\right)$ is a critical point of $J^{\nabla,\alpha_0,l} \Longleftrightarrow$ the (non-random) curve $u(\cdot) \in C^1([0,T]; T_eG)$ coupled with $\alpha(\cdot) \in C^1([0,T]; U^*)$ satisfies the following *semidirect product Euler-Poincaré equation for stochastic particle paths:*

$$\text{(SSEP)} \qquad \begin{cases} \frac{d}{dt}\frac{\delta l}{\delta u} = \mathrm{ad}_u^* \frac{\delta l}{\delta u} + \frac{\delta l}{\delta \alpha} \diamond \alpha + K\left(\frac{\delta l}{\delta u}\right), \\ \frac{d}{dt}\alpha(t) = \frac{1}{2}\sum_{j=1}^{k_2} H_j^2\left(H_j^2 \alpha(t)\right) - u(t)\alpha(t), \end{cases}$$

The first equation of this system is equivalent to the *dissipative Euler-Poincaré variational principle*

$$\delta \int_0^T l(u(t), \alpha(t))dt = 0$$

on $T_eG \times U^*$ defined by $l : T_eG \times U^* \to \mathbb{R}$, for variations of the form

$$\begin{cases} \delta u = \dot{v} + [u, v] + K^*(v), \\ \delta \alpha = -v\alpha \end{cases}$$

where $t \mapsto v(t) \in T_eG$ satisfying $v(0) = 0$, $v(T) = 0$ and $K^* : T_eG \to T_eG$ is the dual map of $K : T_e^*G \to T_e^*G$.

Suppose: ∇ is the Levi-Civita connection of a left invariant metric and $\nabla_{H_i^1} H_i^1 = 0$ for each $i = 1, \ldots, k_1$. Then

$$K^*(u) = -\frac{1}{2} \sum_i \Big(\nabla_{H_i^1} \nabla_{H_i^1} u + \mathbf{R}(u, H_i^1) H_i^1 \Big), \quad \forall u \in \mathfrak{g},$$

where $\mathbf{R}$ is the Riemann curvature tensor.

In addition, if $\{H_i^1 \mid i = 1, \ldots, k_1 = \dim \mathfrak{g}\}$ is an orthonormal basis of $\mathfrak{g}$, then

$$K^*(u) = -\frac{1}{2} \Big(\Delta u + \mathrm{Ric}(u) \Big), \quad \text{where} \quad \Delta u := \Delta U(g)|_{g=e}$$

for the left invariant vector field $U(g) := T_e L_g u$, $\forall u \in \mathfrak{g}$, $g \in G$.

There are right invariant versions of these two theorems, i.e., we consider a right invariant connection ∇ on G and the G-representation on U is on the right. However, the Lie algebra of G is still defined in the standard manner by left invariant vector fields on G.

There are relative sign changes in the equations!

APPLICATION TO MHD

Want to apply the main theorem, right-invariant version, to the "Lie group" Diff$(\mathbb{T}^3)$, a Lie group using the Kriegl-Michor convenient calculus. Can copy the finite dimensional case. Far too complicated. Start with the following SDEs on $G^s :=$ Diff$^s(\mathbb{T}^3)$, $s > 5/2$,

$$\begin{cases} dg_\omega(t,\theta) = \sum_{j=1}^k H_j(g_\omega(t,\theta)) \circ dW_\omega^j(t) + \tilde{u}(t, g_\omega(t,\theta))dt \\ g_\omega(0,\theta) = \theta, \quad \theta \in \mathbb{T}^3, \end{cases}$$

$\tilde{u}(t) := u(t) - \frac{1}{2}\sum_{j=1}^k \nabla^0_{H_j} H_j$, $u \in C^1([0,T]; \mathfrak{X}^s(\mathbb{T}^3))$ is non-random, $dg_\omega(t,\theta)$ is the t-Itô differential, $H_j \in \mathfrak{X}(G^s)$ right invariant extension of $H_1(e) = \sqrt{2\nu}(1,0,0)$, $H_2(e) = \sqrt{2\nu}(0,1,0)$, $H_3(e) = \sqrt{2\nu}(0,0,1)$, $\nu \geq 0$ constant, in the trivialization $T\mathbb{T}^3 = \mathbb{T}^3 \times \mathbb{R}^3$.
Define the process $g^\nu_\omega(t,\theta)$ by

$$\begin{cases} dg^\nu_\omega(t,\theta) = \sqrt{2\nu}dW_\omega(t) + u(t, g^\nu_\omega(t,\theta))dt \\ g^\nu_\omega(0,\theta) = \theta. \end{cases}$$

where $W_\omega(t)$ is a $\mathbb{R}^3$-valued Brownian motion and $dW_\omega(t)$ is the Itô differential of $W_\omega(t)$ with respect to the time variable.

By standard theory of stochastic flows (e.g., Kunita [1990]), if $u \in C^1([0,T]; T_e \operatorname{Diff}^{s'}(\mathbb{T}^3))$ for some $s' > 0$ large enough, then $g_\omega^\nu(t,\cdot) \in T_e \operatorname{Diff}^s(\mathbb{T}^3)$ for every $t \in [0,T]$.

From now on, for simplicity, we always assume u to be regular enough.

U^* some linear space (of functions, densities, differential forms, tensors, etc). The action of G^s on U^* is the pull back map and the action of "Lie algebra" $T_e G^s$ on U^* is the Lie derivative. Let $\alpha_0 \in U^*$, $\tilde{\alpha}(t) := \alpha_0 g_\omega^\nu(t)^{-1}$, and $\alpha(t) := \mathbb{E}\Big[\tilde{\alpha}(t)\Big]$.

What is the analogue of the second equation in (SSEP) with a given $\alpha_0 \in U^*$? This equation is the stochastic version of the evolution of advected quantities.

Start with U^* the space of one-forms on $\mathbb{T}^3$.

Let $g_\omega^\nu(t)$ be given above. Define:

$$\tilde{\alpha}_\omega(t,\theta) := \left(\alpha_0 g_\omega^\nu(t,\cdot)^{-1}\right)(\theta) = \left(\left(g_\omega^\nu(t,\cdot)^{-1}\right)^* \alpha_0\right)(\theta)$$

$$:= \tilde{A}_\omega(t,\theta)\cdot \mathbf{d}\theta := \sum_{i=1}^{3} \tilde{A}_i(t,\theta,\omega)\mathbf{d}\theta_i,$$

where $\left(g_\omega^\nu(t,\cdot)^{-1}\right)^*$ denotes the pull back map by $g_\omega^\nu(t,\cdot)^{-1}$, and

$$\alpha(t,\theta) := \mathbb{E}[\tilde{\alpha}_\omega(t,\theta)] := A(t,\theta)\cdot \mathbf{d}\theta := \sum_{i=1}^{3} A_i(t,\theta)\mathbf{d}\theta_i.$$

Then $\tilde{A}$ satisfies the following SPDE,

$$d\tilde{A}_i(t,\theta,\omega) = -\sum_{j=1}^{3}\sqrt{2\nu}\partial_j\tilde{A}_i(t,\theta,\omega)dW_\omega^j(t) - \sum_{j=1}^{3}\Big(u_j(t,\theta)\partial_j\tilde{A}_i(t,\theta,\omega)$$
$$+\tilde{A}_j(t,\theta,\omega)\partial_i u_j(t,\theta)\Big)dt + \nu\Delta\tilde{A}_i(t,\theta,\omega)dt,\ \ i = 1,2,3,$$

where we use the notation $u(t) := (u_1(t),u_2(t),u_3(t))$; ∂_j and Δ stand for the partial derivative and the Laplacian with respect to the space variable θ of $\tilde{A}_\omega(t,\theta)$, respectively.

This equation can also be expressed as

$$d\tilde{A}_\omega(t,\theta) = -\sqrt{2\nu}\nabla\tilde{A}_\omega(t,\theta)\cdot dW_\omega(t)$$
$$-\Big(u(t,\theta)\times \mathsf{curl}\,\tilde{A}_\omega(t,\theta) - \nabla(u(t,\theta)\cdot\tilde{A}_\omega(t,\theta))\Big)\,dt + \nu\Delta\tilde{A}_\omega(t)dt$$

(the term $d\tilde{A}_\omega(t,\theta)$ above denotes the Itô differential of $\tilde{A}_\omega(t,\theta)$ with respect to the time variable) and we have

$$\begin{cases} \partial_t A(t,\theta) = \Big(u(t,\theta)\times \mathsf{curl}\,A(t,\theta) - \nabla(u(t,\theta)\cdot A(t,\theta))\Big) + \nu\Delta A(t,\theta), \\ A(0,\theta) = A_0(\theta). \end{cases}$$

Comparing formally with the finite dimensional theory:

If $H_1 = \sqrt{2\nu}(1,0,0)$, $H_2 = \sqrt{2\nu}(0,1,0)$, $H_3 = \sqrt{2\nu}(0,0,1)$ and $\alpha(t,\theta) = A(t,\theta)\cdot \mathbf{d}\theta$ we have

$$\nabla_{H_j}H_j = 0, \quad \sum_{j=1}^{3}\alpha(t)H_jH_j = \nu\Delta A(t,\theta)\cdot\mathbf{d}\theta,$$
$$\alpha(t)u(t) = \Big(u(t,\theta)\times \mathsf{curl}\,A(t,\theta) - \nabla(u(t,\theta)\cdot A(t,\theta))\Big)\cdot\mathbf{d}\theta,$$

U^* was a space of one-forms on $\mathbb{T}^3$. Need U^* functions and densities.

• If $\alpha_0 : \mathbb{T}^3 \to \mathbb{R}$ is a C^2 function, then $\alpha(t,\theta)$ satisfies the following transport equation

$$\begin{cases} \partial_t \alpha(t,\theta) = -u(t,\theta)\cdot\nabla\alpha(t,\theta) + \nu\Delta\alpha(t,\theta), \\ \alpha(0,\theta) = \alpha_0(x). \end{cases}$$

• If $\alpha_0 = D_0(\theta)d^3\theta$ is a density (volume form), write $\alpha(t,\theta) = D(t,\theta)d^3\theta$. Then $D(t,\theta)$ satisfies the following forward Kolmogorov (or Fokker-Planck) equation,

$$\begin{cases} \partial_t D(t,\theta) = -\nabla\cdot(Du)(t,\theta) + \nu\Delta D(t,\theta), \\ D(0,\theta) = D_0(\theta), \end{cases}$$

Moreover, if $\alpha_0 = D_0(\theta)d^3\theta$ is a probability measure, let $\tilde{g}^\nu_\omega(t,\theta)$ be the process satisfying

$$d\tilde{g}^\nu_\omega(t,\theta) = \sqrt{2\nu}dW_\omega(t) + u(t,\tilde{g}^\nu_\omega(t,\theta))dt, \quad \tilde{g}^\nu_\omega(0,\theta) = D_0(\theta)d^3\theta$$

Suppose that for every $t\in[0,T]$, the distribution of $\tilde{g}^\nu_\omega(t,\theta)$ is of the form $D(t,\theta)d^3\theta$: then $D(t,\theta)$ satisfies the equation above.

The right-invariant version of the ∇^0-generalized derivative formula holds also in this infinite dimensional case

$$T_{g^\nu_\omega(t,\theta)} R_{g^\nu_\omega(t,\theta)^{-1}} \frac{\mathscr{D}^{\nabla^0} g^\nu_\omega(t,\theta)}{dt} = u(t,\theta).$$

Let $\alpha_0 := (b_0(x), \mathbf{B}_0(\theta) \cdot \mathbf{dS}, D_0(\theta) d^3\theta)$, where $b_0 \in C^2(\mathbb{T}^3)$, $\mathbf{B}_0(\theta) \cdot \mathbf{dS}$ is an exact two-form on $\mathbb{T}^3$, i.e., $\exists$ a one-form $A_0(\theta) \cdot \mathbf{d}\theta$ such that

$$\mathbf{B}_0(\theta) \cdot \mathbf{dS} = \mathbf{d}\Big(A_0(\theta) \cdot \mathbf{d}\theta\Big) = \sum_{1 \le j < k \le 3, i \ne j, i \ne k} \Big(\text{curl}\, A_0(\theta)\Big)_i \mathbf{d}\theta_j \wedge \mathbf{d}\theta_k,$$

$D_0(\theta) d^3\theta$ is a density on $\mathbb{T}^3$.

Let $U^* := \{(b_0(x), \mathbf{B}_0(\theta) \cdot \mathbf{dS}, D_0(\theta) d^3\theta)\}$.

Define $l : T_eG^s \times U^* \to \mathbb{R}$ by

$$l(u, b, \mathbf{B}, D) = \int_{\mathbb{T}^3} \left(\frac{D(\theta)}{2} |u(\theta)|^2 - D(\theta) e(D(\theta), b(\theta)) - \frac{1}{2} |\mathbf{B}(\theta)|^2 \right) d^3\theta,$$

where $u \in T_eG^s = \mathfrak{X}^s(\mathbb{T}^3)$ is the Eulerian (spatial) velocity of the fluid, $b \in C^2(\mathbb{T}^3)$ is the entropy function, $\mathbf{B}(\theta) \cdot \mathbf{dS}$ is an exact differential two-form representing the magnetic field in the fluid, $D(\theta)d^3\theta$ is a density on $\mathbb{T}^3$ representing the mass density of the fluid, and the function $e(D, b)$ is the fluid's specific internal energy. The pressure $p(D, b)$ and the temperature $T(D, b)$ of the fluid are given in terms of a thermodynamic equation of state for the specific internal energy e, namely $\mathbf{d}e = -p\mathbf{d}\left(\frac{1}{D}\right) + T\mathbf{d}b = D^2 \frac{\partial e}{\partial D} \mathbf{d}D + T\mathbf{d}b$. It is assumed that $c^2 := \frac{\partial p}{\partial D} > 0$, where c is the adiabatic sound speed.

Now we would like to apply the main theorem. Cannot, so we derive everything by hand, in analogy with the theorem.

Action functional:

$$\mathbf{J}^{\nabla_0,\alpha_0,l}\left(g_\omega^{\mu_1}, g_\omega^{\mu_2}, g_\omega^{\mu_3}, g_\omega^{\mu_4}\right) := \int_0^T l\left(u(t), b(t), \mathbf{B}(t), D(t)\right) dt,$$

where the stochastic processes $g_\omega^{\mu_i}$, $1 \le i \le 4$, are defined by

$$\begin{cases} dg_\omega^\nu(t,\theta) = \sqrt{2\nu}dW_\omega(t) + u(t, g_\omega^\nu(t,\theta))dt \\ g_\omega^\nu(0,\theta) = \theta, \qquad \nu \ge 0, \end{cases}$$

with $\nu \in \{\mu_i \ge 0 \mid i = 1,2,3,4\}$, the same $u(t,\theta)$, and the curve $t \mapsto (u(t), b(t), \mathbf{B}(t), D(t))$ in U^* is defined by

$$\begin{cases} u(t,\theta) := T_{g_\omega^{\mu_1}(t,\theta)} R_{g_\omega^{\mu_1}(t,\theta)^{-1}} \frac{\mathscr{D}^{\nabla^0} g_\omega^{\mu_1}(t,\theta)}{dt}, \\ b(t,\theta) := \mathbb{E}\left[\left(\left(g_\omega^{\mu_2}(t,\cdot)^{-1}\right)^* b_0\right)(\theta)\right], \\ \mathbf{B}(t,\theta)\cdot \mathbf{dS} := \mathbb{E}\left[\left(\left(g_\omega^{\mu_3}(t,\cdot)^{-1}\right)^* (\mathbf{B}_0(\theta)\cdot \mathbf{dS})\right)(\theta)\right], \\ D(t,\theta)d^3\theta := \mathbb{E}\left[\left(\left(g_\omega^{\mu_4}(t,\cdot)^{-1}\right)^* \left(D_0(\theta)d^3\theta\right)\right)(\theta)\right], \end{cases}$$

where $\left(g_\omega^{\mu_i}(t,\cdot)^{-1}\right)^*$ is the pull back map by $g_\omega^{\mu_i}(t,\cdot)^{-1} : \mathbb{T}^3 \to \mathbb{T}^3$.

For each non-random $v \in C^1([0,1]; T_e G^s)$, $v(0) = v(T) = 0$, regular enough, the perturbation $e_{\varepsilon,v}(t,\cdot) \in G^s$ is given by

$$\begin{cases} \frac{de_{\varepsilon,v}(t,\theta)}{dt} = \varepsilon \frac{\partial}{\partial t} v(t, e_{\varepsilon,v}(t,\theta)) \\ e_{\varepsilon,v}(0,\theta) = \theta \end{cases}$$

$(g_\omega^{\mu_1}, g_\omega^{\mu_2}, g_\omega^{\mu_3}, g_\omega^{\mu_4})$ is a *critical point of $\mathbf{J}^{\nabla^0,\alpha_0,l}$* if $\forall v \in C^1([0,T]; T_e G^s)$ with $v(0) = v(T) = 0$, we have

$$\left. \frac{d\mathbf{J}^{\nabla^0,\alpha_0,l}\left(g_{\varepsilon,v}^{\mu_1}, g_{\varepsilon,v}^{\mu_2}, g_{\varepsilon,v}^{\mu_3}, g_{\varepsilon,v}^{\mu_4}\right)}{d\varepsilon} \right|_{\varepsilon=0} = 0,$$

where $g_{\varepsilon,v}^{\mu_i}(t,\omega)(\theta) := e_{\varepsilon,v}(t, g_\omega^{\mu_i}(t,\theta))$.

The semimartingale $(g_\omega^{\mu_1}, g_\omega^{\mu_2}, g_\omega^{\mu_3}, g_\omega^{\mu_4})$ is a critical point of $\mathbf{J}^{\nabla^0,\alpha_0,l}$ if and only if the following equations hold for $(u(t), b(t), \mathbf{B}(t), D(t))$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \frac{\nabla p}{D} + \frac{\mathbf{B}\times \operatorname{curl}\mathbf{B}}{D} = \mu_1 \Delta u + (\mu_1 - \mu_4)\frac{u\Delta D}{D} + 2\mu_1 \langle \nabla \log D, \nabla u\rangle \\ \partial_t b = -u \cdot \nabla b + \mu_2 \Delta b \\ \partial_t \mathbf{B} = \operatorname{curl}(u \times \mathbf{B}) + \mu_3 \Delta \mathbf{B} \\ \partial_t D = -\nabla \cdot (Du) + \mu_4 \Delta D \\ \nabla \cdot \mathbf{B} = 0, \qquad \text{where } p \text{ is the pressure.} \end{cases}$$

The equation satisfied by D is the analogue of the continuity equation for mass conservation. We have

$$\frac{d}{dt}\int D d\theta = \int [-\nabla \cdot (Du) + \mu_4 \Delta D] d\theta = 0.$$

The equation for the density contains the Laplacian, so it is different from the deterministic case, because the underlying paths $g_\omega^\nu(t)$ are now random, with Brownian diffusion coefficient of intensity given by the viscosity. Thus we have, for all regular functions f on $\mathbb{T}^3$,

$$\frac{d}{dt}\mathbb{E}\left[\int_{\mathbb{T}^3} f(g_\omega^\nu(t,\theta))d^3\theta\right] = \mathbb{E}\left[\int_{\mathbb{T}^3} ((u\cdot\nabla f) + \nu\Delta f)\,(g_\omega^\nu(t,\theta))d^3\theta\right],$$

which gives precisely the Fokker-Planck equation for D.

The reason for choosing processes g^{μ_i} with different constants μ_i is that the viscosity constants are different.

Furthermore, if we take $\mu_1 = \mu_4 = \nu$ for some $\nu > 0$, we get,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \frac{1}{D}\nabla p + \frac{1}{D}\mathbf{B} \times \mathrm{curl}\,\mathbf{B} = \nu\Delta u + 2\nu\langle \nabla \log D, \nabla u\rangle \\ \partial_t b = -u \cdot \nabla b + \mu_2 \Delta b \\ \partial_t \mathbf{B} = \mathrm{curl}(u \times \mathbf{B}) + \mu_3 \Delta \mathbf{B} \\ \partial_t D = -\nabla \cdot (Du) + \nu \Delta D \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

where u is the fluid velocity, b is the entropy, $\mathbf{B}$ is the magnetic field, p is the pressure, D is the mass density, and ν, μ_3, and μ_2 are the constants of viscosity, resistivity, and diffusivity, respectively.

The energy decays along the solutions of this system. For this, the presence of the term $2\nu\langle \nabla \log D, \nabla u\rangle$ is important.

In particular, if we take $D(t) = 1$, $b(t) = 1$ for every $t \in [0, T]$ in the system above, we obtain the following viscous MHD equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p + \mathbf{B} \times \operatorname{curl} \mathbf{B} = \nu \Delta u \\ \partial_t \mathbf{B} = \operatorname{curl}(u \times \mathbf{B}) + \mu_3 \Delta \mathbf{B} \\ \nabla \cdot u = 0, \ \nabla \cdot \mathbf{B} = 0. \end{cases}$$